

A FAMILY OF SYMMETRIC MIXED FINITE ELEMENTS FOR LINEAR ELASTICITY ON TETRAHEDRAL GRIDS

JUN HU AND SHANGYOU ZHANG

ABSTRACT. A family of stable mixed finite elements for the linear elasticity on tetrahedral grids are constructed, where the stress is approximated by symmetric $H(\text{div})$ - P_k polynomial tensors and the displacement is approximated by C^{-1} - P_{k-1} polynomial vectors, for all $k \geq 4$. Numerical tests are provided.

Keywords. mixed finite element, symmetric finite element, linear elasticity, conforming finite element, tetrahedral grids, inf-sup condition.

AMS subject classifications. 65N30, 73C02.

1. INTRODUCTION

In the Hellinger-Reissner mixed formulation of the linear elasticity equations, the stress is sought in $H(\text{div}, \Omega, \mathbb{S})$ and the displacement in $L^2(\Omega, \mathbb{R}^3)$. It is a challenge to design stable mixed finites mainly due to the symmetric constraint of the stress tensor \mathbb{S} . To overcome this difficulty, ealier works adopted composite element techniques or weakly symmetric methods, cf. [3, 6, 7, 25, 27, 29, 30, 31]. In [9], Arnold and Winther designed the first family of mixed finite element methods in 2D, based on polynomial shape function spaces. From then on, various stable mixed elements have been constructed, see [2, 4, 5, 9, 11, 17, 10, 19, 22, 26, 32, 33, 8, 12, 18, 20, 21, 24, 23].

As the displacement function is in $L^2(\Omega, \mathbb{R}^3)$, a natural discretization is the piecewise P_{k-1} polynomial without interelement continuity. It is a long-standing and challenging problem if the stress tensor can be discretized by an appropriate P_k finite element subspace of $H(\text{div}, \Omega, \mathbb{S})$. Adams and Cockburn constructed such a mixed finite element in [2] where the discrete stress space is the space of $H(\text{div}, \Omega, \mathbb{S})$ - P_{k+2} tenors whose divergence is a P_{k-1} polynomial on each tetrahedron, for $k = 2$. The method was modified and extended to a family of elements, $k \geq 2$, by Arnold, Awanou and Winther [5]. Mathematically speaking, these methods are two-order suboptimal. In this paper, we solve this open problem by constructing a suitable $H(\text{div}, \Omega, \mathbb{S})$ - P_k , instead of above P_{k+2} , finite element space for the stress discretization, for $k \geq 4$. In these elements, the symmetric stress tensor is approximated by the full C^0 - P_k space enriched by some so-called $H(\text{div})$ edge-bubble functions locally on each tetrahedron. A new way of proof is developed to establish the stability of the mixed elements, by characterizing the divergence of local stress space. This space of divergence of local stress space is exactly the subspace of P_k displacements orthogonal to the local rigid-motion. The optimal order error estimate is proved, verified by numerical tests of P_4 and P_5 mixed elements.

The rest of the paper is organized as follows. In Section 2, we define the weak problem and the finite element method. In section 3, we prove the well-posedness of the finite element problem, i.e. the discrete coerciveness and the discrete inf-sup condition. By which, the optimal order convergence of the new element follows. In Section 4, we provide some numerical results, using P_4 and P_5 finite elements.

2. THE FAMILY OF FINITE ELEMENTS

Based on the Hellinger-Reissner principle, the linear elasticity problem within a stress-displacement $(\sigma-u)$ form reads: Find $(\sigma, u) \in \Sigma \times V := H(\text{div}, \Omega, \mathbb{S} = \text{symmetric } \mathbb{R}^{3 \times 3}) \times L^2(\Omega, \mathbb{R}^3)$, such that

$$(2.1) \quad \begin{cases} (A\sigma, \tau) + (\text{div}\tau, u) = 0 & \text{for all } \tau \in \Sigma, \\ (\text{div}\sigma, v) = (f, v) & \text{for all } v \in V. \end{cases}$$

Here the symmetric tensor space for stress Σ and the space for vector displacement V are, respectively,

$$(2.2) \quad H(\text{div}, \Omega, \mathbb{S}) := \left\{ \sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \in H(\text{div}, \Omega) \mid \sigma^T = \sigma, \right\},$$

$$(2.3) \quad L^2(\Omega, \mathbb{R}^3) := \left\{ (u_1 \ u_2 \ u_3)^T \mid u_i \in L^2(\Omega) \right\}.$$

This paper denotes by $H^k(T, X)$ the Sobolev space consisting of functions with domain $T \subset \mathbb{R}^3$, taking values in the finite-dimensional vector space X , and with all derivatives of order at most k square-integrable. For our purposes, the range space X will be either \mathbb{S} , \mathbb{R}^3 , or \mathbb{R} . $\|\cdot\|_{k,T}$ is the norm of $H^k(T)$. \mathbb{S} denotes the space of symmetric tensors, $H(\text{div}, T, \mathbb{S})$ consists of square-integrable symmetric matrix fields with square-integrable divergence. The $H(\text{div})$ norm is defined by

$$\|\tau\|_{H(\text{div}, T)}^2 := \|\tau\|_{L^2(T)}^2 + \|\text{div}\tau\|_{L^2(T)}^2.$$

$L^2(T, \mathbb{R}^3)$ is the space of vector-valued functions which are square-integrable. Here, the compliance tensor $A = A(x) : \mathbb{S} \rightarrow \mathbb{S}$, characterizing the properties of the material, is bounded and symmetric positive definite uniformly for $x \in \Omega$.

This paper deals with a pure displacement problem (2.1) with the homogeneous boundary condition that $u \equiv 0$ on $\partial\Omega$. But the method and the analysis work for mixed boundary value problems and the pure traction problem.

The domain Ω is subdivided by a family of quasi-uniform tetrahedral grids \mathcal{T}_h (with the grid size h). We introduce the finite element space of order k ($k \geq 4$) on \mathcal{T}_h . The displacement space is the full C^{-1} - P_{k-1} space

$$(2.4) \quad V_h = \{v \in L^2(\Omega, \mathbb{R}^3) \mid v|_K \in P_{k-1}(K, \mathbb{R}^3) \text{ for all } K \in \mathcal{T}_h\}.$$

The discrete stress space of order k ($k \geq 4$) is defined abstractly as

$$(2.5) \quad \Sigma_h = \left\{ \sigma \in H(\text{div}, \Omega, \mathbb{S}) \mid \sigma|_K \in P_k(K, \mathbb{S}) \ \forall K \in \mathcal{T}_h, \right. \\ \left. \sigma|_K(v_i) = \sigma|_{K'}(v_i) \ \forall v_i \in \mathcal{V}_h \text{ and } v_i \in K \cap K' \right\},$$

where \mathcal{V}_h is the set of vertices of the tetrahedral grid \mathcal{T}_h , and v_i is a common vertex of tetrahedra K and K' . Computationally, for building a basis for Σ_h , we need to

give another definition of Σ_h . Σ_h is a $H(\text{div})$ bubble enrichment of the H^1 space

$$(2.6) \quad \tilde{\Sigma}_h = \left\{ \sigma \in H^1(\Omega, \mathbb{S}) \mid \sigma|_K \in P_k(K, \mathbb{S}) \ \forall K \in \mathcal{T}_h \right\}.$$

In computation, we still uses $6 \times \dim P_k$ Lagrange nodal basis locally on each tetrahedron K , i.e., the standard basis for H^1 finite element space $\tilde{\Sigma}_h$. But globally, roughly speaking, we break each of $(k-1) = \dim P_{k-2,1D}$ zero-flux (on all six edges) edge-bubble functions into n_0 basis functions, where n_0 tetrahedra share this common edge, cf. Figure 2.1, and break each of $(k-2)(k-1)/2 = \dim P_{k-3,2D}$ zero-flux (on all four face triangles) facee-bubble functions into 2 basis functions, on the two tetrahedra sharing a common face triangle. Here, on each triangle, we have three sets of non-zero edge-bubble functions enriched, all of which have a zero-flux on the triangle. To avoid too much technical details, we only define the local edge-bubble functions, but we do not discuss on eliminating linearly dependent bubbles (with H^1 - P_k basis functions).

Tetrahedron K :

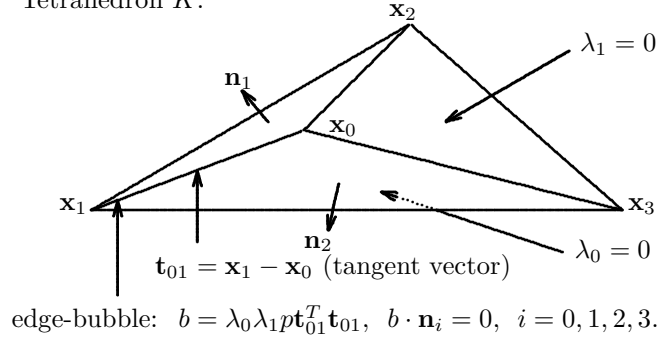


FIGURE 2.1. An edge-bubble function $b = \lambda_0 \lambda_1 p \mathbf{t}_{01}^T \mathbf{t}_{01}$, $p \in P_{k-2}(K)$, on an edge $\mathbf{x}_0\mathbf{x}_1$ of tetrahedron K .

Let $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2$ and \mathbf{x}_3 be the four vertices of a tetrahedron K , cf. Figure 2.1. The referencing mapping is then

$$\mathbf{x} = F_K(\hat{\mathbf{x}}) = \mathbf{x}_0 + \begin{pmatrix} \mathbf{x}_1 - \mathbf{x}_0 & \mathbf{x}_2 - \mathbf{x}_0 & \mathbf{x}_3 - \mathbf{x}_0 \end{pmatrix} \hat{\mathbf{x}},$$

mapping the reference tetrahedron $\hat{K} = \{0 \leq \hat{x}_1, \hat{x}_2, \hat{x}_3, 1 - \hat{x}_1 - \hat{x}_2 - \hat{x}_3 \leq 1\}$ to K . Then the inverse mapping is

$$(2.7) \quad \hat{\mathbf{x}} = \begin{pmatrix} \mathbf{n}_1^T \\ \mathbf{n}_2^T \\ \mathbf{n}_3^T \end{pmatrix} (\mathbf{x} - \mathbf{x}_0),$$

where

$$(2.8) \quad \begin{pmatrix} \mathbf{n}_1^T \\ \mathbf{n}_2^T \\ \mathbf{n}_3^T \end{pmatrix} = \begin{pmatrix} \mathbf{x}_1 - \mathbf{x}_0 & \mathbf{x}_2 - \mathbf{x}_0 & \mathbf{x}_3 - \mathbf{x}_0 \end{pmatrix}^{-1}.$$

By (2.7), these normal vectors are coefficients of the barycentric variables:

$$\begin{aligned}\lambda_1 &= \mathbf{n}_1 \cdot (\mathbf{x} - \mathbf{x}_0), \\ \lambda_2 &= \mathbf{n}_2 \cdot (\mathbf{x} - \mathbf{x}_0), \\ \lambda_3 &= \mathbf{n}_3 \cdot (\mathbf{x} - \mathbf{x}_0), \\ \lambda_0 &= 1 - \lambda_1 - \lambda_2 - \lambda_3.\end{aligned}$$

On each face triangle, say $\mathbf{x}_0\mathbf{x}_2\mathbf{x}_3$, all three edges (the tangent vector), $\mathbf{x}_0\mathbf{x}_2$, $\mathbf{x}_0\mathbf{x}_3$ and $\mathbf{x}_2\mathbf{x}_3$, are orthogonal to the face normal vector \mathbf{n}_1 . For convenience, we introduce the tangent vectors and their tensors:

$$\begin{aligned}(2.9) \quad \mathbf{t}_{01} &= \mathbf{x}_1 - \mathbf{x}_0, & T_{01} &= \mathbf{t}_{01}^T \mathbf{t}_{01}, \\ \mathbf{t}_{02} &= \mathbf{x}_2 - \mathbf{x}_0, & T_{02} &= \mathbf{t}_{02}^T \mathbf{t}_{02}, \\ \mathbf{t}_{03} &= \mathbf{x}_3 - \mathbf{x}_0, & T_{03} &= \mathbf{t}_{03}^T \mathbf{t}_{03}, \\ \mathbf{t}_{12} &= \mathbf{x}_2 - \mathbf{x}_1, & T_{12} &= \mathbf{t}_{12}^T \mathbf{t}_{12}, \\ \mathbf{t}_{23} &= \mathbf{x}_3 - \mathbf{x}_2, & T_{23} &= \mathbf{t}_{23}^T \mathbf{t}_{23}, \\ \mathbf{t}_{13} &= \mathbf{x}_3 - \mathbf{x}_1, & T_{13} &= \mathbf{t}_{13}^T \mathbf{t}_{13}.\end{aligned}$$

With them, we define the $H(\text{div}, K, \mathbb{S})$ bubble functions

$$(2.10) \quad \Sigma_{K,b} = \text{span}\{\lambda_0\lambda_1p_1T_{01}, \lambda_0\lambda_2p_2T_{02}, \lambda_0\lambda_3p_3T_{03}, \\ \lambda_1\lambda_2p_4T_{12}, \lambda_2\lambda_3p_5T_{23}, \lambda_1\lambda_3p_6T_{13}\},$$

where p_1, \dots, p_6 are 3D P_{k-2} polynomials. Note that each bubble function, say, $\lambda_0\lambda_1p_1T_{01}$, vanishes on two face triangles ($\lambda_0 = 0, \lambda_1 = 0$) and has zero normal component on the other two face triangles ($T_{01} \cdot \mathbf{n}_2 = \mathbf{0}, T_{01} \cdot \mathbf{n}_3 = \mathbf{0}$.) Thus, the matching of $\text{div } \tau_h$ and v_h is done locally on K , independently of the matching on neighboring elements. To characterize the bubble space $\Sigma_{K,b}$, we need the following lemma.

Lemma 2.1. *The six symmetric tensors T_{ij} in (2.9) are linearly independent, and form a basis of \mathbb{S} .*

Proof. Each tensor $T_{ij} = \mathbf{t}_{ij}\mathbf{t}_{ij}^T$ is a positive semi-definite matrix, on a tetrahedron K . We would show that the constants c_{ij} are all equal to zero in

$$T = c_{01}T_{01} + c_{02}T_{02} + c_{03}T_{03} + c_{12}T_{12} + c_{23}T_{23} + c_{13}T_{13} = \mathbf{0}.$$

First, we compute the bilinear form, cf. Figure 2.1, by (2.8),

$$\mathbf{n}_1^T T \mathbf{n}_1 = c_{01}1 \cdot 1 + c_{02}0 + c_{03}0 + c_{12}(-1)(-1) + c_{23}0 + c_{13}(-1)(-1) = 0.$$

Here, by (2.8) and (2.9),

$$\begin{aligned}\mathbf{t}_{01}^T \mathbf{n}_1 &= 1, \\ \mathbf{t}_{12}^T \mathbf{n}_1 &= (\mathbf{t}_{02}^T - \mathbf{t}_{01}^T) \mathbf{n}_1 = 0 - 1, \\ \mathbf{t}_{13}^T \mathbf{n}_1 &= (\mathbf{t}_{03}^T - \mathbf{t}_{01}^T) \mathbf{n}_1 = 0 - 1.\end{aligned}$$

Symmetrically, by evaluating $\mathbf{n}_i T \mathbf{n}_i$ for $i = 0, 1, 2, 3$, where $\mathbf{n}_0 = -\mathbf{n}_1 - \mathbf{n}_2 - \mathbf{n}_3$, we have

$$(2.11) \quad \begin{cases} c_{01} + c_{02} + c_{03} = 0, \\ c_{01} + c_{12} + c_{13} = 0, \\ c_{02} + c_{12} + c_{23} = 0, \\ c_{03} + c_{13} + c_{23} = 0. \end{cases}$$

Note that $\mathbf{n}_0 \neq \mathbf{0}$ as K is a non-singular tetrahedron. Next, we introduce three (non-unit) vectors \mathbf{s}_i orthogonal to the three pairs of skew edges, $\overline{\mathbf{x}_0 \mathbf{x}_1}$ and $\overline{\mathbf{x}_2 \mathbf{x}_3}$, $\overline{\mathbf{x}_0 \mathbf{x}_2}$ and $\overline{\mathbf{x}_1 \mathbf{x}_3}$, $\overline{\mathbf{x}_0 \mathbf{x}_3}$ and $\overline{\mathbf{x}_1 \mathbf{x}_2}$, respectively, cf. Figure 2.1. That is,

$$\mathbf{s}_1 = \frac{\mathbf{t}_{01} \times \mathbf{t}_{23}}{6|K|},$$

because $|K| \neq 0$ and consequently $|\mathbf{t}_{01} \times \mathbf{t}_{23}| \neq 0$. Thus $\mathbf{s}_1 \cdot \mathbf{t}_{01} = 0$, $\mathbf{s}_1 \cdot \mathbf{t}_{02} = -1$, $\mathbf{s}_1 \cdot \mathbf{t}_{03} = -1$, $\mathbf{s}_1 \cdot \mathbf{t}_{12} = -1$, $\mathbf{s}_1 \cdot \mathbf{t}_{13} = -1$, and $\mathbf{s}_1 \cdot \mathbf{t}_{23} = 0$. By evaluating $\mathbf{s}_i^T T \mathbf{s}_i$, it follows that

$$(2.12) \quad \begin{cases} c_{02} + c_{03} + c_{12} + c_{13} = 0, \\ c_{01} + c_{03} + c_{12} + c_{23} = 0, \\ c_{01} + c_{02} + c_{13} + c_{23} = 0. \end{cases}$$

By the first two equations in (2.11) and the first equation in (2.12), we get

$$2c_{01} = 0.$$

Symmetrically, we find all $c_{ij} = 0$. Thus $\{T_{ij}\}$ is a linearly independent set of tensors. As $\dim \mathbb{S} = 6$, $\{T_{ij}\}$ is a basis. \blacksquare

An equivalent but more practical definition of the stress finite element space Σ_h is

$$(2.13) \quad \Sigma_h = \left\{ \sigma = \sigma_a + \sigma_b \in H(\operatorname{div}, \Omega, \mathbb{S}) \mid \sigma_a \in \tilde{\Sigma}_h, \sigma_b|_K \in \Sigma_{K,b} \ \forall K \in \mathcal{T}_h \right\},$$

where $\tilde{\Sigma}_h$ and $\Sigma_{K,b}$ are defined in (2.6) and (2.10), respectively.

It follows from the definition of V_h (P_{k-1} polynomials) and Σ_h (P_k polynomials) that

$$\operatorname{div} \Sigma_h \subset V_h.$$

This, in turn, leads to a strong divergence-free space:

$$(2.14) \quad \begin{aligned} Z_h &:= \{ \tau_h \in \Sigma_h \mid (\operatorname{div} \tau_h, v) = 0 \text{ for all } v \in V_h \} \\ &= \{ \tau_h \in \Sigma_h \mid \operatorname{div} \tau_h = 0 \text{ pointwise} \}. \end{aligned}$$

The mixed finite element approximation of Problem (1.1) reads: Find $(\sigma_h, u_h) \in \Sigma_h \times V_h$ such that

$$(2.15) \quad \begin{cases} (A\sigma_h, \tau) + (\operatorname{div} \tau, u_h) = 0 & \text{for all } \tau \in \Sigma_h, \\ (\operatorname{div} \sigma_h, v) = (f, v) & \text{for all } v \in V_h. \end{cases}$$

3. STABILITY AND CONVERGENCE

The convergence of the finite element solutions follows the stability and the standard approximation property. So we consider first the well-posedness of the discrete problem (2.15). By the standard theory, we only need to prove the following two conditions, based on their counterpart at the continuous level.

- (1) K-ellipticity. There exists a constant $C > 0$, independent of the meshsize h such that

$$(3.1) \quad (A\tau, \tau) \geq C \|\tau\|_{H(\text{div})}^2 \quad \text{for all } \tau \in Z_h,$$

where Z_h is the divergence-free space defined in (2.14).

- (2) Discrete B-B condition. There exists a positive constant $C > 0$ independent of the meshsize h , such that

$$(3.2) \quad \inf_{0 \neq v \in V_h} \sup_{0 \neq \tau \in \Sigma_h} \frac{(\text{div} \tau, v)}{\|\tau\|_{H(\text{div})} \|v\|_{L^2(\Omega)}} \geq C.$$

It follows from $\text{div} \Sigma_h \subset V_h$ that $\text{div} \tau = 0$ for any $\tau \in Z_h$. This implies the above K-ellipticity condition (3.1). It remains to show the discrete B-B condition (3.2), in the following two lemmas.

Lemma 3.1. *For any $v_h \in V_h$, there is a $\tau_h \in \tilde{\Sigma}_h \subset \Sigma_h$ such that, for all polynomial $p \in P_{k-3}(K, \mathbb{R}^3)$, $K \in \mathcal{T}_h$,*

$$(3.3) \quad \int_K (\text{div} \tau_h - v_h) \cdot p \, d\mathbf{x} = 0 \quad \text{and} \quad \|\tau_h\|_{H(\text{div})} \leq C \|v_h\|_{L^2(\Omega)}.$$

Proof. Let $v_h \in V_h$. By the stability of the continuous formulation, cf. [9], there is a $\tau \in H^1(\Omega, \mathbb{S})$ such that,

$$\text{div} \tau = v_h \quad \text{and} \quad \|\tau\|_{H^1(\Omega)} \leq C \|v_h\|_{L^2(\Omega)}.$$

As $\tau \in H^1(\Omega, \mathbb{S})$, we modify the Scott-Zhang [28] interpolation operator slightly to define a flux preserving interpolation.

$$I_h : H^1(\Omega, \mathbb{S}) \rightarrow \Sigma_h \cap H^1(\Omega, \mathbb{S}) = \tilde{\Sigma}_h, \\ \tau \mapsto \tau_h =: I_h \tau.$$

Here the interpolation is done inside a subspace, the continuous finite element subspace $\Sigma_h \cap H^1(\Omega, \mathbb{S})$. $I_h \tau$ is defined by its values at the Lagrange nodes.

At a vertex node or a node inside an edge, \mathbf{x}_i , $I_h \tau(\mathbf{x}_i)$ is defined as the nodal value of τ at the point if τ is continuous, but in general, $I_h \tau(\mathbf{x}_i)$ is defined as an average value on a face triangle, on whose edge the node is, as in [28]. After defining the nodal values at edges of tetrahedra, the nodal values of τ_h at the nodes inside each face triangle T of a tetrahedron are defined by the L^2 -orthogonal projection on the triangle T :

$$(3.4) \quad \int_T \tau_{h,ij} p \, dS = \int_T \tau_{ij} p \, dS \quad \forall p \in P_{k-3}(T, \mathbb{R}),$$

$i, j = 1, 2, 3$, where $\tau_{h,ij}$ and τ_{ij} are the (i, j) -th components of τ_h and τ , respectively, and T is a face triangle of a tetrahedron in the tetrahedral triangulation \mathcal{T}_h . The number of equations in (3.4) is same as the number of internal degrees of

freedom of P_k polynomials, $\dim P_{k-3}$. At the Lagrange nodes inside a tetrahedron, $I_h \tau(\mathbf{x}_i)$ is defined by the L^2 -orthogonal projection on the tetrahedron, satisfying

$$(3.5) \quad \int_K \tau_{h,ij} p \, d\mathbf{x} = \int_K \tau_{ij} p \, d\mathbf{x} \quad \forall p \in P_{k-4}(K, \mathbb{R}),$$

where K is an element of \mathcal{T}_h . It follows by the stability of the Scott-Zhang operator that

$$\|I_h \tau\|_{H^1(\Omega)} \leq C \|\tau\|_{H^1(\Omega)} \leq C \|v_h\|_{L^2(\Omega)}.$$

In particular,

$$\|I_h \tau\|_{H(\text{div})} \leq \|I_h \tau\|_{H^1(\Omega)} \leq C \|v_h\|_{L^2(\Omega)}.$$

By (3.4) and (3.5), we get a partial-divergence matching property of I_h : for any $p \in P_{k-3}(K, \mathbb{R}^3)$, as the symmetric gradient $\epsilon(p) \in P_{k-4}(K, \mathbb{S})$,

$$\begin{aligned} \int_K (\text{div } \tau_h - v_h) \cdot p \, d\mathbf{x} &= \int_{\partial K} (\tau_h \mathbf{n}) \cdot p \, ds - \int_K \tau_h : \epsilon(p) \, d\mathbf{x} - \int_K v_h \cdot p \, d\mathbf{x} \\ &= \int_{\partial K} (\tau \mathbf{n}) \cdot p \, ds - \int_K \tau : \epsilon(p) \, d\mathbf{x} - \int_K v_h \cdot p \, d\mathbf{x} \\ &= \int_K (\text{div } \tau - v_h) \cdot p \, d\mathbf{x} = 0. \end{aligned}$$

■

Lemma 3.2. *For any $v_h \in V_h$, if*

$$(3.6) \quad \int_K v_h \cdot p \, d\mathbf{x} = 0 \quad \text{for all } p \in P_{k-3}(K, \mathbb{R}^3) \text{ and all } K \in \mathcal{T}_h,$$

there is a $\tau_h \in \Sigma_h$ such that

$$(3.7) \quad \text{div } \tau_h = v_h \quad \text{and} \quad \|\tau_h\|_{H(\text{div})} \leq C \|v_h\|_{L^2(\Omega)}.$$

Proof. As we assume polynomial degree $k \geq 4$ in V_h , $p \in P_{k-3}(K, \mathbb{R}^3) \supset P_1(K, \mathbb{R}^3) \supset R(K)$ where $R(K)$ is the set of 6-dimensional, local rigid motions:

$$(3.8) \quad R(K) = \left\{ \begin{pmatrix} a_1 - a_4 y - a_5 z \\ a_2 + a_4 x - a_6 z \\ a_3 + a_5 x + a_6 y \end{pmatrix} \mid a_1, a_2, a_3, a_4, a_5, a_6 \in \mathbb{R} \right\}.$$

So if v_h satisfies (3.6), v_h is in the following local rigid-motion free space:

$$(3.9) \quad V_{h,\perp R} = \left\{ v_h \in V_h \mid \int_K v_h \cdot p \, d\mathbf{x} = 0 \, \forall p \in R(K) \text{ and } \forall K \in \mathcal{T}_h \right\}.$$

We will prove a stronger result that if $v_h \in V_{h,\perp R}$, then there is a τ_h satisfying (3.7). This τ_h is constructed, according to v_h , on each element K , independently of the construction on neighboring elements. On one element K , we show $\text{div } \Sigma_{K,b} = V_{h,\perp R}|_K$ where $\Sigma_{K,b}$ is the edge-bubble space, defined in (2.10). If $\text{div } \Sigma_{K,b} \neq V_{h,\perp R}|_K$, there is a nonzero $v_h \in V_{h,\perp R}$ such that

$$\int_K \text{div } \tau_h \cdot v_h \, d\mathbf{x} = 0 \quad \forall \tau_h \in \Sigma_{K,b}.$$

By integration by parts, for $\tau_h \in \Sigma_{K,b}$, we have

$$(3.10) \quad \int_K \text{div } \tau_h \cdot v_h \, d\mathbf{x} = \int_K \tau_h : \epsilon(v_h) \, d\mathbf{x} = 0,$$

where $\epsilon(v_h)$ is the symmetric gradient, $(\nabla v_h + \nabla^T v_h)/2$.

Let $\{M_{ij}, i = 0, 1, 2, j = i, \dots, 3\}$ be the dual basis of the symmetric space, under \mathbb{R}^9 inner-product, of $\{T_{ij}\}$, defined in (2.9), i.e.

$$(3.11) \quad M_{ij} = M_{ij}^T, \quad M_{ij} \cdot T_{i'j'} = \delta_{ij,i'j'}.$$

For example, if K is the unit right tetrahedron, then $\{T_{ij}\}$ would be

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix},$$

and the unique $\{M_{ij}\}$ would be

$$\begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 0 & 0 \\ 1/2 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1/2 & 0 \\ -1/2 & 1 & 1/2 \\ 0 & 1/2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1/2 \\ 0 & 0 & 1/2 \\ 1/2 & 1/2 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & -1/2 & 0 \\ -1/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & -1/2 \\ 0 & 0 & 0 \\ -1/2 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1/2 \\ 0 & -1/2 & 0 \end{pmatrix}.$$

Under the dual basis, we have a unique expansion, as $\epsilon(v_h) \in P_{k-2}(K, \mathbb{S})$,

$$(3.12) \quad \epsilon(v_h) = q_1 M_{01} + q_2 M_{02} + q_3 M_{03} q_4 M_{12} + q_5 M_{23} + q_6 M_{13},$$

for some $q_i \in P_{k-2}(K)$. Selecting $\tau_1 = \lambda_0 \lambda_1 q_1 T_{01} \in \Sigma_{K,b}$, we have, by (3.11),

$$0 = \int_K \tau_1 : \epsilon(v_h) d\mathbf{x} = \int_K \lambda_0 \lambda_1 q_1^2(\mathbf{x}) d\mathbf{x}.$$

As $\lambda_0 \lambda_1 > 0$ on K , we conclude that $q_1 \equiv 0$. Similarly, the other five q_i in (3.12) are zero. Thus, by (3.10), $v_h \equiv 0$ and $\text{div } \sigma_{K,b} = V_{h,\perp R}|_K$. As the matching $\text{div } \tau_h = v_h$ is done on one element K , by affine mapping and scaling argument, (3.7) holds. \blacksquare

We are in the position to show the well-posedness of the discrete problem.

Lemma 3.3. *For the discrete problem (2.15), the K -ellipticity (3.1) and the discrete B-B condition (3.2) hold uniformly. Consequently, the discrete mixed problem (2.15) has a unique solution $(\sigma_h, u_h) \in \Sigma_h \times V_h$.*

Proof. The K -ellipticity immediately follows from the fact that $\text{div } \Sigma_h \subset V_h$. To prove the discrete B-B condition (3.2), for any $v_h \in V_h$, it follows from Lemma 3.1 that there exists a $\tau_1 \in \Sigma_h$ such that, for any polynomial $p \in P_{k-3}(K, \mathbb{R}^3)$,

$$(3.13) \quad \int_K (\text{div } \tau_1 - v_h) \cdot p d\mathbf{x} = 0 \quad \text{and} \quad \|\tau_1\|_{H(\text{div})} \leq C \|v_h\|_{L^2(\Omega)}.$$

Then it follows from Lemma 3.2 that there is a $\tau_2 \in \Sigma_h$ such that

$$(3.14) \quad \text{div } \tau_2 = v_h - \text{div } \tau_1 \quad \text{and} \quad \|\tau_2\|_{H(\text{div})} \leq C \|\text{div } \tau_1 - v_h\|_{L^2(\Omega)},$$

Let $\tau = \tau_1 + \tau_2$. This implies that

$$(3.15) \quad \text{div } \tau = v_h \quad \text{and} \quad \|\tau\|_{H(\text{div})} \leq C \|v_h\|_{L^2(\Omega)},$$

this proves the discrete B-B condition (3.2). \blacksquare

Theorem 3.1. *Let $(\sigma, u) \in \Sigma \times V$ be the exact solution of problem (2.1) and $(\tau_h, u_h) \in \Sigma_h \times V_h$ the finite element solution of (2.15). Then, for $k \geq 4$,*

$$(3.16) \quad \|\sigma - \sigma_h\|_{H(\text{div})} + \|u - u_h\|_{L^2(\Omega)} \leq Ch^k (\|\sigma\|_{H^{k+1}(\Omega)} + \|u\|_{H^k(\Omega)}).$$

Proof. The stability of the elements and the standard theory of mixed finite element methods [13, 14] give the following quasioptimal error estimate immediately

$$(3.17) \quad \|\sigma - \sigma_h\|_{H(\text{div})} + \|u - u_h\|_{L^2(\Omega)} \leq C \inf_{\tau_h \in \Sigma_h, v_h \in V_h} (\|\sigma - \tau_h\|_{H(\text{div})} + \|u - v_h\|_{L^2(\Omega)}).$$

Let P_h denote the local L^2 projection operator, or triangle-wise interpolation operator, from V to V_h , satisfying the error estimate

$$(3.18) \quad \|v - P_h v\|_{L^2(\Omega)} \leq Ch^k \|v\|_{H^k(\Omega)} \text{ for any } v \in H^k(\Omega, \mathbb{R}^3).$$

Choosing $\tau_h = I_h \sigma \in \Sigma_h$ where I_h is defined in (3.4) and (3.5), we have [28], as I_h preserves symmetric P_k functions locally,

$$(3.19) \quad \|\sigma - \tau_h\|_{L^2(\Omega)} + h \|\sigma - \tau_h\|_{H(\text{div})} \leq Ch^{k+1} \|\sigma\|_{H^{k+1}(\Omega)}.$$

Let $v_h = P_h v$ and $\tau_h = I_h \sigma$ in (3.17), by (3.18) and (3.19), we obtain (3.16). ■

4. NUMERICAL TESTS

We compute one example in 3D, by P_4 and by P_5 mixed finite element methods. It is a pure displacement problem on the unit cube $\Omega = (0, 1)^3$ with a homogeneous boundary condition that $u \equiv 0$ on $\partial\Omega$. In the computation, we let

$$A\sigma = \frac{1}{2\mu} \left(\sigma - \frac{\lambda}{2\mu + n\lambda} \text{tr}(\sigma)\delta \right), \quad n = 3,$$

where $\delta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and $\mu = 1/2$ and $\lambda = 1$ are the Lamé constants.

Let the exact solution on the unit square $[0, 1]^3$ be

$$(4.1) \quad u = \begin{pmatrix} 2^4 \\ 2^5 \\ 2^6 \end{pmatrix} x(1-x)y(1-y)z(1-z).$$

Then, the true stress function σ and the load function f are defined by the equations in (2.1), for the given solution u .

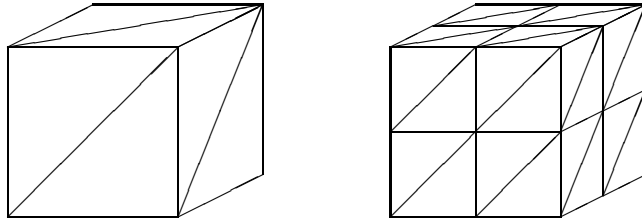


FIGURE 4.1. The initial grid for (4.1), and its level 2 refinement.

In the computation, the level one grid is the given domain with a diagonal line shown in Figure 4.1. Each grid is refined into a half-sized grid uniformly, to get a

higher level grid, shown in Figure 4.1. In all the computation, the discrete systems of equations are solved by Matlab backslash solver. In Table 4.1, the errors and the convergence order in various norms are listed for the true solution (4.1), by P_4 mixed finite elements in (2.5) and (2.4), with $k = 4$ there. Here I_h is the usual nodal interpolation operator. The optimal order of convergence is achieved in Table 4.1, confirming Theorem 3.1.

TABLE 4.1. The error and the order of convergence by P_4 finite elements, $k = 4$ in (2.4) and (2.5), for (4.1).

	$\ I_h\sigma - \sigma_h\ _{L^2(\Omega)}$	h^n	$\ I_h u - u_h\ _{L^2(\Omega)}$	h^n	$\ \operatorname{div}(I_h\sigma - \sigma_h)\ _{L^2(\Omega)}$	h^n
1	0.33567012	0.0	0.05860521	0.0	3.41111411	0.0
2	0.02041247	4.0	0.00661542	3.1	0.21319463	4.0
3	0.00125425	4.0	0.00044841	3.9	0.01332466	4.0

In Table 4.2, the errors and the convergence order in various norms are listed for the true solution (4.1), by P_5 mixed finite elements in (2.5) and (2.4), with $k = 5$ there. Here the exact solution σ is a polynomial tensor of degree 5. Thus, it is in the stress finite element space Σ_h and the finite element solution σ_h is exact. It is computed so, shown in the second column and the six column in Table 4.2. The optimal order of convergence is achieved for the displacement u in Table 4.2 (up to the computer accuracy), confirming Theorem 3.1.

TABLE 4.2. The error and the order of convergence by P_5 finite elements, $k = 5$ in (2.4) and (2.5), for (4.1).

	$\ I_h\sigma - \sigma_h\ _{L^2(\Omega)}$	h^n	$\ I_h u - u_h\ _{L^2(\Omega)}$	h^n	$\ \operatorname{div}(I_h\sigma - \sigma_h)\ _{L^2(\Omega)}$	h^n
1	0.00000002	0.0	0.01937914	0.0	0.00000011	0.0
2	0.00000002	0.0	0.00089726	4.4	0.00000031	0.0

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LMAM AND SCHOOL OF MATHEMATICAL SCIENCES, PEKING UNIVERSITY, BEIJING 100871, P. R. CHINA. HUJUN@MATH.PKU.EDU.CN

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF DELAWARE, NEWARK, DE 19716, USA. SZHANG@UDEL.EDU